Rate of Escape of Random Walks on Regular Languages and Free Products by Amalgamation of Finite Groups

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We consider random walks on the set of all words over a finite alphabet such that in each step only the last two letters of the current word may be modified and only one letter may be adjoined or deleted. We assume that the transition probabilities depend only on the last two letters of the current word. Furthermore, we consider also the special case of random walks on free products by amalgamation of finite groups which arise in a natural way from random walks on the single factors. The aim of this paper is to compute several equivalent formulas for the rate of escape with respect to natural length functions for these random walks using different techniques.

Keywords: Random Walks, Regular Languages, Free Products by Amalgamation, Rate of Escape

1 Introduction

Let A be a finite alphabet and let A^* be the set of all finite words over the alphabet A, where ε is the empty word. Furthermore, let $l : A \to [0; \infty)$ be a function representing a 'letter length'. The extension of l to A^* defined by $l(a_1 \dots a_n) = \sum_{i=1}^n l(a_i)$ gives then a suitable 'word length'. We consider a transient Markov chain $(X_n)_{n \in \mathbb{N}_0}$ on A^* with $X_0 = \varepsilon$ such that transition probabilities depend only on the last two letters of the actual word and in each step only the last two letters may be modified and only one letter may be adjoined or deleted. We are interested in whether the sequence of random variables $l(X_n)/n$ converges almost surely to a constant, and if so, to compute this constant. If the limit exists, it is called the *rate of escape*, or the *drift with respect to* l. In this paper, we study this question for random walks on regular languages and on free products by amalgamation of finite groups, which form special cases of regular languages and are a generalization of free products of groups.

It is well-known that the rate of escape w.r.t. the natural word length exists for a random walk on a finitely generated group, which is governed by a probability measure on the group. This follows from *Kingman's subadditive ergodic theorem*; see Kingman (13), Derriennic (3) and Guivarc'h (10). There are many detailed results for random walks on free products by amalgamation: Picardello and Woess (20) showed that a locally compact free product by amalgamation of compact groups acts naturally on

a tree. They also derived the behaviour of the n-step transition probabilities. Cartwright and Soardi (2) investigated random walks on free products by amalgamation, where the amalgamating subgroup is finite and normal. They derived a formula for the Green function $G(z) = \sum_{n \geq 0} p^{(n)}(e,e)z^n$, where $p^{(n)}(e,e)$ is the n-step return probability from the identity e, of the random walk on the amalgamated product in terms of the Green functions of the single factors that is essentially the same as in Woess (23). For random walks on free products of finite groups Mairesse and Mathéus (16) have developed a specific technique for the computation of the rate of escape. For this purpose, they have to solve a more elegant system of algebraic equations than we have to solve, but our results will be more general. Three different formulas for the rate of escape of random walks on free products of graphs and groups are derived in Gilch (9). The techniques used in (9) were the starting point for the computation of the rate of escape in this paper. An important link between drifts and harmonic analysis was obtained by Varopoulos (22). He proved that for symmetric finite range random walks on groups the existence of non-trivial bounded harmonic functions is equivalent to a non-zero rate of escape. The recent work of Karlsson and Ledrappier (12) generalizes this result to random walks with finite first moment of the step lengths. This leads to a link between the rate of escape and the entropy of random walks; compare e.g. with Kaimanovich and Vershik (11) and Erschler (6).

We also consider random walks on regular languages which can be seen as a generalization of free products by amalgamation. Random Walks on this class of structures have been investigated by several authors: Malyshev (17), (18) and Gairat, Malyshev, Menshikov, Pelikh (8) stated criteria for transience, null-recurrence and positive recurrence. Moreover, Malyshev proved limit theorems concerning existence of the stationary distribution and speed in the transient case and convergence of conditional distributions in the ergodic case; in particular, he showed that the rate of escape w.r.t. the natural word length (that is, $l(\cdot) = 1$) is constant and it is strictly positive if and only if the random walk is transient. Yambartsev and Zamyatin (25) proved limit theorems for random walks on two semi-infinite strings over a finite alphabet. Lalley (14) also investigated random walks on regular languages. He found out that the n-step return probabilities must obey one of three different types of power laws. His analysis is based on a finite algebraic system of generating functions related to the Green function. This algebraic system is also used in this paper to compute explicit formulas for the rate of escape. The rate of escape has also been studied on trees, which may be seen as a special case of our context: Nagnibeda and Woess (19, Section 5) proved that the rate of escape of random walks on trees with finitely many cone types is non-zero and give a formula for it. One of the techniques used in this paper for the computation of the rate of escape was motivated by Nagnibeda and Woess.

Our aim is to compute formulas for the rate of escape of random walks on regular languages and free products by amalgamation of finite groups. In Section 2 we compute the rate of escape of random walks on regular languages, while in Section 3 we compute it for random walks on free products by amalgamation. In Section 3.2 we compute the rate of escape analogously to Section 2.2 and in Section 3.3 we compute it by an application of a theorem of Sawyer and Steger (21). In Section 3.4 we use the algebraic group structure of free products by amalgamation to compute the rate of escape with respect to the natural word length. This approach is based on a technique which was already used by Ledrappier (15) and Furstenberg (7). Finally, in Section 4 we give sample computations.

2 Rate of Escape of Random Walks on Regular Languages

2.1 Regular Languages and Random Walks

Let A be a finite alphabet and ε be the empty word. A *random walk on a regular language* is a Markov chain on the set $A^* := \bigcup_{n\geq 1} A^n \cup \{\varepsilon\}$ of all finite words over the alphabet A, whose transition probabilities obey the following rules:

(i) Only the last two letters of the current word may be modified.

(ii) Only one letter may be adjoined or deleted at one instant of time.

(iii) Adjunction and deletion may only be done at the end of the current word.

(iv) Probabilities of modification, adjunction or deletion depend only on the last two letters of the current word.

Compare with Lalley (14). The hypothesis that transition probabilities depend only on the last two letters of the current word can be weakened to dependence of the last $K \geq 2$ letters by a "recoding trick", which is also described by Lalley. In general, a regular language is a subset of A^* whose words are accepted by a finite-state automaton. It is necessary that by each modification of a word of the regular language in one single step a new word of the regular language is created. The results below, however, are so general such that w.l.o.g. – for ease and better readability – we may assume that the regular language consists of the whole set A^*.

The random walk on A^* is described by the sequence of random variables $(X_n)_{n\in\mathbb{N}_0}$. Initially, we have $X_0 = \varepsilon$. For two words $w, w' \in A^*$ we write ww' for the concatenated word. We use the following abbreviations for the transition probabilities: for $w \in A^*$, $a, a', b \in A$, $b', c' \in A \cup \{\varepsilon\}$, $n \in \mathbb{N}_0$, let be

$$\mathbb{P}[X_{n+1} = wa'b'c' \mid X_n = wab] = p(ab, a'b'c'),$$
$$\mathbb{P}[X_{n+1} = b'c' \mid X_n = a] = p(a, b'c'),$$
$$\mathbb{P}[X_{n+1} = b' \mid X_n = \varepsilon] = p(\varepsilon, b').$$

If we want to start the random walk at $w \in A^*$ instead of ε, we write for short $\mathbb{P}_w[\cdot] := \mathbb{P}[\cdot \mid X_0 = w]$. Suppose we are given a function $l : A \to [0;\infty)$. We extend l to A^* by defining $l(a_1a_2\dots a_n) := \sum_{i=1}^{n} l(a_i)$ for $a_1a_2\dots a_n \in A^n$. Additionally, we set $l(\varepsilon) := 0$. If $l(a) = 1$ for each $a \in A$, then l is just the *natural word length* which is denoted by $|\cdot|$. If there is a non-negative constant ℓ such that

$$\lim_{n\to\infty} \frac{l(X_n)}{n} = \ell \quad \text{almost surely,}$$

then ℓ is called the *rate of escape* with respect to l. Malyshev (17) proved that the rate of escape w.r.t. the natural word length exists. Furthermore, by Malyshev follows that the rate of escape w.r.t. l is zero if and only if $(X_n)_{n\in\mathbb{N}_0}$ is recurrent. Our aim is to compute a formula for ℓ in the transient case. Therefore, we assume from now on transience of $(X_n)_{n\in\mathbb{N}_0}$.

Moreover, we assume that the random walk on A^* is *suffix-irreducible*, that is, for all $w \in A^*$ with $\mathbb{P}[X_m = w] > 0$ for some $m \in \mathbb{N}$ and for all $ab \in A^2$ there is some $n \in \mathbb{N}$ such that

$$\mathbb{P}\Big[\exists w_1 \in A^* : X_n = ww_1ab, \forall k < n : |X_k| \geq |w| \,\Big|\, X_0 = w\Big] > 0.$$

If suffix-irreducibility is dropped, then the rate of escape may be non-deterministic; e.g., if $A = \{a, b\}$ with $l(a) = l(b) = 1$ and $p(aa, aaa) = p > 1/2$, $p(aa, a) = p(a, \varepsilon) = 1 - p$, $p(\varepsilon, a) = p(\varepsilon, b) = 1/2$, $p(bb, bbb) = q > 1/2$, $p(bb, b) = p(b, \varepsilon) = 1 - q$ with $p \neq q$, then $l(X_n)/n$ converges only non-deterministically.

2.2   The Rate of Escape

The technique we use to compute ℓ was motivated by Nagnibeda and Woess (19, Section 5). For $k \in \mathbb{N}_0$ we define the *k-th exit time* as

$$\mathbf{e}_k := \sup\{m \in \mathbb{N}_0 \,\big|\, |X_m| = k\}.$$

As the alphabet A is finite and the random walk on A^* is assumed to be transient, we have $\mathbf{e}_k < \infty$ almost surely for every $k \in \mathbb{N}_0$. Furthermore, we write $\mathbf{W}_k := X_{\mathbf{e}_k}$ and $\mathbf{i}_k := \mathbf{e}_k - \mathbf{e}_{k-1}$ with $\mathbf{e}_{-1} := 0$. We show at first that $(\mathbf{W}_k, \mathbf{i}_k)_{k \geq 3}$ is a Markov chain. For this purpose, we introduce some useful functions: for $a, b, c \in A$ and real $z > 0$ define

$$H(ab, c|z) \quad := \quad \sum_{n=1}^{\infty} \mathbb{P}_{ab}\big[X_n = c, \forall m < n : |X_m| > 1\big]\, z^n,$$

$$\xi(abc) \quad := \quad \sum_{a'b'c' \in A^3} p(bc, a'b'c') \cdot \Big(1 - \sum_{d \in A} H(b'c', d|1)\Big).$$

Observe that

$$\mathbb{P}_{abc}\big[X_n = ab', \forall m < n : |X_m| > 2\big] = \mathbb{P}_{bc}\big[X_n = b', \forall m < n : |X_m| > 1\big],$$

as the transition probabilities depend only on the last two letters of the current word and in each step only one letter may be deleted. Thus, the number $\xi(abc)$ is the probability of starting at $abc \in A^3$ such that $|X_n| \geq 4$ for all $n \geq 1$, and it does not depend on the letter "a". Furthermore, let be $[a_1 \ldots a_n]_3 := a_{n-2}a_{n-1}a_n$, if $a_1 \ldots a_n \in A^*$ with $n \geq 3$. With this notation we get:

Proposition 2.1 *The stochastic process* $(\mathbf{W}_k, \mathbf{i}_k)_{k \geq 3}$ *is a Markov chain with transition probabilities*

$$\mathbb{P}\big[\mathbf{W}_{k+1} = x_{k+1}, \mathbf{i}_{k+1} = n_{k+1} \,\big|\, \mathbf{W}_k = x_k, \mathbf{i}_k = n_k\big]$$

$$= \quad \frac{\xi([x_{k+1}]_3)}{\xi([x_k]_3)} \cdot \mathbb{P}_{x_k}\big[X_{n_{k+1}} = x_{k+1}, \forall i \in \{1, \ldots, n_{k+1}\} : |X_i| > k\big]$$

for $n_k, n_{k+1} \in \mathbb{N}$, $x_k, x_{k+1} \in A^*$ *with* $|x_k| = k$, $|x_{k+1}| = k + 1$ *and* $\mathbb{P}\big[\mathbf{W}_k = x_k, \mathbf{i}_k = n_k\big] > 0$.

Proof: Let be $n_0, n_1, \ldots, n_{k+1} \in \mathbb{N}$ and $x_0, x_1, \ldots, x_{k+1} \in A^*$ with $|x_j| = j$ for $j \in \{0, 1, \ldots, k+1\}$. Define the event

$$\big[\mathbf{W}_0^m = x_0^m, \mathbf{i}_0^m = n_0^m\big] := \big[\forall j \in \{0, 1, \ldots, m\} : \mathbf{W}_j = x_j, \mathbf{i}_j = n_j\big],$$

where $m \in \{k, k+1\}$. With this notation we get

$$\mathbb{P}\big[\mathbf{W}_0^k = x_0^k, \mathbf{i}_0^k = n_0^k\big] \quad = \quad \mathbb{P}\left[\begin{array}{c} \forall j \in \{0, \ldots, k\}\, \forall \lambda \in \{0, \ldots, n_j\} : \\ |X_{n_1 + \cdots + n_{j-1} + \lambda}| \geq j, X_{n_1 + \cdots + n_j} = x_j \end{array}\right] \cdot \mathbb{P}_{x_k}\big[\forall n \geq 1 : |X_n| > k\big]$$

$$= \quad \mathbb{P}\left[\begin{array}{c} \forall j \in \{0, \ldots, k\}\, \forall \lambda \in \{0, \ldots, n_j\} : \\ |X_{n_1 + \cdots + n_{j-1} + \lambda}| \geq j, X_{n_1 + \cdots + n_j} = x_j \end{array}\right] \cdot \xi([x_k]_3).$$

Analogously,

$$\mathbb{P}\big[\mathbf{W}_0^{k+1} = x_0^{k+1}, \mathbf{i}_0^{k+1} = n_0^{k+1}\big]$$

$$= \quad \mathbb{P}\left[\begin{array}{c} \forall j \in \{0,\dots,k\}\,\forall \lambda \in \{0,\dots,n_j\}: \\ |X_{n_1+\cdots+n_{j-1}+\lambda}| \geq j, X_{n_1+\cdots+n_j} = x_j \end{array}\right]$$

$$\cdot\mathbb{P}_{x_k}\big[\forall i \in \{1,\dots,n_{k+1}\}: |X_i| > k, X_{n_{k+1}} = x_{k+1}\big] \cdot \xi([x_{k+1}]_3).$$

Thus, under the assumption that $\mathbb{P}\big[\mathbf{W}_0^k = x_0^k, \mathbf{i}_0^k = n_0^k\big] > 0$ we obtain

$$\mathbb{P}\big[\mathbf{W}_0^{k+1} = x_0^{k+1}, \mathbf{i}_0^{k+1} = n_0^{k+1} \,\big|\, \mathbf{W}_0^k = x_0^k, \mathbf{i}_0^k = n_0^k\big]$$

$$= \quad \frac{\xi([x_{k+1}]_3)}{\xi([x_k]_3)} \cdot \mathbb{P}_{x_k}\big[\forall i \in \{1,\dots,n_{k+1}\}: |X_i| > k, X_{n_{k+1}} = x_{k+1}\big].$$

$$\square$$

Observe that $\mathbb{P}_{x_k}\big[\forall i \in \{1,\dots,n_{k+1}\}: |X_i| > k, X_{n_{k+1}} = x_{k+1}\big]$ depends only on n_{k+1}, $[x_k]_3$ and $[x_{k+1}]_3$. We use this observation to construct a new Markov chain on the state space

$$\mathcal{Z} := \big\{(abc, n) \in \overline{A}^3 \times \mathbb{N} \,\big|\, \exists de \in A^2 : \mathbb{P}_{de}[X_n = abc, \forall m \in \{1,\dots,n\} : |X_m| > 2]\big\},$$

where $\overline{A}^3 := \{abc \in A^3 \mid \xi(abc) > 0\}$ with the following transition probabilities:

$$q\big((abc, n),(a'b'c', n')\big) = \frac{\xi(a'b'c')}{\xi(abc)} \cdot \mathbb{P}_{abc}\big[X_n = aa'b'c', \forall i \in \{1,\dots,n'\} : |X_i| \geq 4\big].$$

Observe that

$$\mathbb{P}\big[\mathbf{W}_{k+1} = x_{k+1}, \mathbf{i}_{k+1} = n_{k+1} \,\big|\, \mathbf{W}_k = x_k, \mathbf{i}_k = n_k\big] = q\big(([x_k]_3, n_k),([x_{k+1}]_3, n_{k+1})\big)$$

for $k \geq 3$ and that the transition probabilities do not depend on n_k. This provides that also $\big([\mathbf{W}_k]_3\big)_{k\geq 3}$ is a Markov chain on $\overline{A}^3$ with transition probabilities

$$\tilde{q}(abc, a'b'c') = \sum_{n' \in \mathbb{N}} q\big((abc, n_{abc}),(a'b'c', n')\big),$$

where the n_{abc}'s on the right hand side of the equation may be chosen arbitrarily. Observe that $[\mathbf{W}_k]_3$ may only take a finite number of states, since the alphabet A is finite and $|[\mathbf{W}_k]_3| = 3$. At this point we need the above made assumption of suffix-irreducibility; this provides that $\big([\mathbf{W}_k]_3\big)_{k\geq 3}$ is irreducible and therefore has an invariant probability measure ν.

Lemma 2.2 *Let be $abc \in A^3$ and $n \in \mathbb{N}$ and define*

$$\pi(abc, n) := \sum_{def \in \overline{A}^3} \nu(def)\, q\big((def, n_{def}),(abc, n)\big),$$

where n_{def} can be chosen arbitrarily. Then π is the unique invariant probability measure of $\big([\mathbf{W}_k]_3, \mathbf{i}_k\big)_{k\geq 3}$.

Proof: It is a straightforward computation to prove the lemma:

$$\sum_{(ghi,s)\in\mathcal{Z}} \pi(ghi,s)\, q\big((ghi,s),(abc,n)\big)$$

$$= \sum_{(ghi,s)\in\mathcal{Z}}\sum_{def\in\overline{A}^3} \nu(def)\, q\big((def,n_{def}),(ghi,s)\big)\, q\big((ghi,s),(abc,n)\big)$$

$$= \sum_{ghi\in\overline{A}^3} q\big((ghi,n_{ghi}),(abc,n)\big) \sum_{def\in\overline{A}^3} \nu(def) \sum_{s\in\mathbb{N}} q\big((def,n_{def}),(ghi,s)\big)$$

$$= \sum_{ghi\in\overline{A}^3} q\big((ghi,n_{ghi}),(abc,n)\big)\, \nu(ghi) = \pi(abc,n).$$

$\square$

Define $g : \mathcal{Z} \to \mathbb{N} : (abc,n) \mapsto n$. An application of the *ergodic theorem for positive recurrent Markov chains* yields

$$\frac{1}{k}\sum_{i=3}^{k} g\big([\mathbf{W}_k]_3,\mathbf{i}_k\big) = \frac{\mathbf{e}_k - \mathbf{e}_2}{k} \xrightarrow{k\to\infty} \int g(abc,n)\, d\pi \quad \text{almost surely,} \tag{2.1}$$

if the integral exists. Our next aim is to ensure finiteness of this integal and to compute a formula for it. For this purpose, we define

$$\overline{G}(ab,cd|z) \;\; := \;\; \sum_{n=0}^{\infty} \mathbb{P}_{ab}\big[X_n = cd, \forall m \leq n : |X_m| \geq 2\big]\, z^n,$$

$$\mathcal{K}(ab,cde|z) \;\; := \;\; \sum_{n=1}^{\infty} \mathbb{P}_{ab}\big[X_n = cde, \forall m \in \{1,\ldots,n\} : |X_m| \geq 3\big]\, z^n$$

$$= \;\; \sum_{fg\in A^2} p(ab,cfg)\cdot z\cdot \overline{G}(fg,de|z),$$

where $a,b,c,d,e \in A$ and $z > 0$. We have the following linear system of equations:

$$\overline{G}(ab,cd|z) \;\; = \;\; \delta_{ab}(cd) + \sum_{c'd'\in A^2} p(ab,c'd')\cdot z\cdot \overline{G}(c'd',cd|z) \, +$$

$$+ \sum_{c'd'e'\in A^3} p(ab,c'd'e')\cdot z\cdot \sum_{f'\in A} H(d'e',f'|z)\cdot \overline{G}(c'f',cd|z). \tag{2.2}$$

Moreover, we also have the following finite system of equations:

$$H(ab,c|z) \;\; = \;\; p(ab,c)\cdot z + \sum_{de\in A^2} p(ab,de)\cdot z\cdot H(de,c|z)$$

$$+ \sum_{def\in A^3} p(ab,def)\cdot z\cdot \sum_{g\in A} H(ef,g|z)\cdot H(dg,c|z); \tag{2.3}$$

compare with Lalley (14). The system (2.3) consists of equations of quadratic order, and thus the functions $H(\cdot,\cdot|z)$ are algebraic, if the transition probabilities are algebraic. If one has solved this system, then the linear system of equations (2.2) can be solved easily. In particular, the functions $\overline{G}(\cdot,\cdot|z)$ are also algebraic for algebraic transition probabilities. Observe that we can write

$$\tilde{q}(abc, a'b'c') = \frac{\xi(a'b'c')}{\xi(abc)}\, \mathcal{K}(bc, a'b'c'|1),$$

providing ν can be computed if (2.3) can be solved. Turning back to our integral in (2.1) we can now compute:

Proposition 2.3 *We have* $\lim_{k\to\infty} \mathbf{e}_k/k = \Lambda$ *almost surely, where*

$$\Lambda := \sum_{abc,def\in\overline{A}^3} \nu(def)\cdot\frac{\xi(abc)}{\xi(def)}\cdot\frac{\partial}{\partial z}\left[\sum_{gh\in A^2} p(ef, agh)\cdot z\cdot \overline{G}(gh, bc|z)\right]\Bigg|_{z=1}.$$

Proof: We compute straight-forward:

$$\int g(abc, n)\, d\pi$$

$$= \sum_{(abc,n)\in\mathcal{Z}} n\cdot \sum_{def\in\overline{A}^3} \nu(def)\cdot q\big((def, n_{def}), (abc, n)\big)$$

$$= \sum_{def\in\overline{A}^3} \nu(def) \sum_{(abc,n)\in\mathcal{Z}} n\cdot\frac{\xi(abc)}{\xi(def)}\cdot \mathbb{P}_{def}\big[X_n = dabc, \forall m\in\{1,\dots,n\} : |X_m|\geq 4\big]$$

$$= \sum_{abc,def\in\overline{A}^3} \nu(def)\cdot\frac{\xi(abc)}{\xi(def)}\cdot \sum_{n\in\mathbb{N}} n\cdot \mathbb{P}_{def}\big[X_n = dabc, \forall m\in\{1,\dots,n\} : |X_m|\geq 4\big]$$

$$= \sum_{abc,def\in\overline{A}^3} \nu(def)\cdot\frac{\xi(abc)}{\xi(def)}\cdot \frac{\partial}{\partial z}\big[\mathcal{K}(ef, abc|z)\big]\Big|_{z=1}.$$

Finiteness of the integal is ensured if all functions $H(\cdot,\cdot|z)$ and $\overline{G}(\cdot,\cdot|z)$ have radii of convergence bigger than 1. But this follows from Lalley (14): he proved that the Green functions of random walks on regular languages have radii of convergence bigger than 1. $\qquad\square$

Now we can state an explicit formula for the rate of escape:

Theorem 2.4 *There is some non-negative constant* ℓ *such that*

$$\ell = \lim_{n\to\infty}\frac{l(X_n)}{n} = \frac{\Delta}{\Lambda} > 0 \quad \textit{almost surely,}$$

where

$$\Delta := \sum_{abc,def\in\overline{A}^3} \nu(def)\, l(a)\,\frac{\xi(abc)}{\xi(def)}\mathcal{K}(ef, abc|1).$$

In particular, $\lim_{n\to\infty}|X_n|/n = 1/\Lambda$ *almost surely.*

Proof: With $h : \mathcal{Z} \to \mathbb{N}$ defined by $h(abc, n) := l(a)$ we obtain

$$\frac{1}{n}\sum_{k=3}^{n} h\big([\mathbf{W}_k]_3, \mathbf{i}_k\big) \xrightarrow{n \to \infty} \int h \, d\pi = \lim_{m \to \infty} \frac{l(X_{\mathbf{e}_m})}{m}.$$

Simple computations lead to the following formula for this limit:

$$\Delta := \int h \, d\pi = \sum_{abc,def \in \overline{A}^3} \nu(def) \cdot l(a) \cdot \frac{\xi(abc)}{\xi(def)} \cdot \mathcal{K}(ef, abc|1).$$

Defining $\mathbf{k}(n) := \max\{k \in \mathbb{N}_0 \mid \mathbf{e}_k \leq n\}$ we obtain analogously to Nagnibeda and Woess (19, Proof of Theorem D)

$$\ell = \lim_{n \to \infty} \frac{l(X_n)}{n} = \lim_{n \to \infty} \frac{l(X_{\mathbf{e}_{\mathbf{k}(n)}})}{\mathbf{k}(n)} \frac{\mathbf{k}(n)}{\mathbf{e}_{\mathbf{k}(n)}} = \frac{\Delta}{\Lambda} > 0.$$

$\square$

Observe that for algebraic transition probabilities the rate of escape is obtained by solving the algebraic system of equations (2.3). This yields that the rate of escape is also algebraic, if the transition probabilities are algebraic and $l(\cdot)$ takes only algebraic values.

3 Rate of Escape of Random Walks on Free Products by Amalgamation

In this section we compute three formulas for the rate of escape of random walks on free products by amalgamation of finite groups. This class of structures form special cases of regular languages.

3.1 Free Products by Amalgamation

Let be $2 \leq r \in \mathbb{N}$. Consider finite groups $\Gamma_1, \dots, \Gamma_r$ with identities $e_1, \dots, e_r$ and subgroups $H_1 \subset \Gamma_1$, $\dots, H_r \subset \Gamma_r$. We assume that $H_1, \dots, H_r$ are isomorphic, that is, there is a finite group H such that there are isomorphisms $\varphi_1 : H \to H_1, \dots, \varphi_r : H \to H_r$. Thus, we identify in the following each H_i with H. To explain the concept of free products by amalgamation, we give at first a simple example: consider $\Gamma_1 = \Gamma_2 = \mathbb{Z}/d\mathbb{Z}$, $d \in \mathbb{N}$ even, and the subgroup $H = \mathbb{Z}/2\mathbb{Z}$. Let Γ_1 be generated by an element a, and Γ_2 by an element b. The free product by amalgamation $\mathbb{Z}/d\mathbb{Z} *_{\mathbb{Z}/2\mathbb{Z}} \mathbb{Z}/d\mathbb{Z}$ consists then of all finite words over the alphabet $\{a, b\}$, where we have the relations $a^{d/2} = b^{d/2}$ and $a^d = b^d = \varepsilon$. That is, any two words which can be deduced from each other with these relations represent the same element. The relation $a^{d/2} = b^{d/2}$ means that the subgroup $\mathbb{Z}/2\mathbb{Z}$ in both copies of $\mathbb{Z}/d\mathbb{Z}$ are identified. E.g., for $d = 4$ it is $a^3 bab^2 = ab^3 a^3 = aba$. To help visualize the concept of free products by amalgamation, we may also think of the Cayley graphs X_i of Γ_i. We connect the graphs X_i by identifying the subgroups $H = H_i$; at each non-trivial coset of H in all graphs X_i we attach copies of X_j, $j \neq i$, where the coset is identified with H of the copy of X_j. This construction is then iterated.

We explain below free products by amalgamation in more detail. The quotient Γ_i/H consists of all sets of sets $yH = \{yh \mid h \in H\}$, where $y \in \Gamma_i$. We fix representatives $x_{i,1} = e_i, x_{i,2}, \dots, x_{i,n_i}$ for the elements of Γ_i/H, that is, for each $y \in \Gamma_i$ there is a unique $x_{i,k}$ with $y \in x_{i,k}H$. We write $\Gamma_i^\times := \Gamma_i \setminus H$

and $R_i := \{x_{i,2}, \ldots, x_{i,n_i}\}$ with $n_i = [\Gamma_i : H]$. For any element $x \in \bigcup_{i=1}^{r} \Gamma_i$ we set $\tau(x) := i$, if $x \in \Gamma_i^{\times}$, and $\tau(x) := 0$, if $x \in H$.

The free product of $\Gamma_1, \ldots, \Gamma_r$ by amalgamation with respect to H is given by

$$\Gamma := \Gamma_1 *_H \Gamma_2 *_H \cdots *_H \Gamma_r,$$

which consists of all finite words of the form

$$x_1 x_2 \ldots x_n h, \tag{3.1}$$

where $h \in H$, $n \in \mathbb{N}_0$ and $x_1, \ldots, x_n \in \bigcup_{i=1}^{r} R_i$ such that $\tau(x_i) \neq \tau(x_{i+1})$. In the following we will always use this representation of words. Suppose we are given a function $l : \bigcup_{i=1}^{r} R_i \to [0; \infty)$. Then we extend l to a length function on Γ by setting $l(x_1 \ldots x_n h) := \sum_{i=1}^{r} l(x_i)$. The *natural word length* is defined to be $\|x_1 \ldots x_n h\| := n$. In particular, $l(h) = \|h\| = 0$ for all $h \in H$. For two words $w_1 = x_1 x_2 \ldots x_m h$, $w_2 = y_1 y_2 \ldots y_n h' \in \Gamma$ a group operation is defined in the following way: first, concatenate the two words, then replace hy_1 in the middle by $y_1' h_1$ such that y_1' is a representative for the class of hy_1. Iterate the last step with $h_1 y_2$ and so on. Finally, we get a word of the form $x_1 \ldots x_n y_1' \ldots y_n' h_n$ with $h_n \in H$, that is, we get the requested equivalent form (3.1) for the concatenated word $w_1 \circ w_2$. Note also that $w^{-1} = h^{-1} x_m^{-1} \ldots x_1^{-1}$ is the inverse of w_1 and can be written in the form of (3.1). The empty word e is the identity of this group operation. Observe that each Γ_i is a subset of Γ.

Suppose we are given probability measures μ_i on Γ_i. Let $\alpha_1, \ldots, \alpha_r$ be strictly positive real numbers such that $\sum_{i=1}^{r} \alpha_i = 1$. A probability measure on Γ is given by

$$\mu(x) := \begin{cases} \alpha_{\tau(x)} \mu_{\tau(x)}(x), & \text{if } x \in \bigcup_{i=1}^{r} \Gamma_i^{\times} \\ \sum_{i=1}^{r} \alpha_i \mu_i(x), & \text{if } x \in H \\ 0, & \text{otherwise} \end{cases}.$$

The n-th convolution power of μ is denoted by $\mu^{(n)}$. The random walk $(X_n)_{n \in \mathbb{N}_0}$ on Γ is then governed by the transition probabilities $p(w_1, w_2) := \mu(w_1^{-1} w_2)$, where $w_1, w_2 \in \Gamma$. Initially, $X_0 := e$.

Lemma 3.1 *The random walk on Γ is recurrent if and only if $r = 2 = [\Gamma_1 : H] = [\Gamma_2 : H]$.*

Proof: Assume $r = 2 = [\Gamma_1 : H] = [\Gamma_2 : H]$. This provides $H \trianglelefteq \Gamma_1, \Gamma_2$, that is, $(\Gamma_1 * \Gamma_2)/H \simeq (\Gamma_1/H) * (\Gamma_2/H)$ and $\Gamma_1/H \simeq \mathbb{Z}/2\mathbb{Z} \simeq \Gamma_2/H$. Since it is well-known that each random walk on the free product $(\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/2\mathbb{Z})$, which arises from a convex combination of probability measures on the single factors, is recurrent, the random walk on Γ also must be recurrent.

Assume now that $r = 2 = [\Gamma_1 : H] = [\Gamma_2 : H]$ does not hold. Then either $r \geq 3$ or w.l.o.g. $[\Gamma_1 : H] \geq 3$. In both cases, Γ is non-amenable (for further details see e.g. Woess (24, Th.10.10)). With Woess (24, Cor.12.5) we get that the random walk on Γ must be transient. $\square$

From now on we exclude the case $r = 2 = [\Gamma_1 : H_1] = [\Gamma_2 : H_2]$. In the following three subsections we want to compute three explicit formulas for the rate of escape of our random walk on Γ. The first approach uses the technique from the previous section, while the second approach arises from an application of a theorem of Sawyer and Steger (21). The third technique uses the group structure of Γ, but is restricted to the computation of the rate of escape w.r.t. the natural word length.

3.2 Exit Time Technique

We use the technique developped in Section 2.2 to compute ℓ. Notice that Γ is a special case of a regular language and our random walk on Γ fulfills the assumptions of our investigated random walks on regular languages: starting from a word $x_1 \ldots x_n h \in \Gamma$ we can only move in one step with positive probability to a word of the form

- $x_1 \ldots x_{n-1} x'_n h'$ with $x'_n h' \in \Gamma^\times_{\tau(x_n)}$, namely with probability $\mu(h^{-1} x_n^{-1} x'_n h')$, or

- $x_1 \ldots x_n x_{n+1} h'$ with $x_{n+1} h' \in \bigcup_{i=1, i \neq \tau(x_n)}^r \Gamma^\times_i$, namely with probability $\mu(h^{-1} x_{n+1} h')$, or

- $x_1 \ldots x_{n-1} h'$ with $h' \in H$, namely with probability $\mu(h^{-1} x_n^{-1} h')$,

where $x_1, \ldots, x_{n+1}, x'_n \in \bigcup_{i=1}^r R_i$ and $h, h' \in H$.

We may now apply the technique of Section 2.2 with some slight modifications and simplifications. The exit-times are now given by

$$\mathbf{e}_k := \sup\{m \in \mathbb{N}_0 \mid \|X_m\| = k\}.$$

Analogously, $\mathbf{W}_k := X_{\mathbf{e}_k}$ and $\mathbf{i}_k := \mathbf{e}_k - \mathbf{e}_{k-1}$. We define for any $x, y \in R_i$, $i \in \{1, \ldots, r\}$, $h, h' \in H$,

$$H(xh, h'|z) \quad := \quad \sum_{n=1}^\infty \mathbb{P}_{xh}\big[X_n = h', \forall m < n : \|X_m\| \geq 1\big] z^n,$$

$$\xi(i) \quad := \quad \sum_{gh_1 \in \bigcup_{j=1, j \neq i}^r \Gamma^\times_j} \mu(gh_1) \cdot \Big(1 - \sum_{h_2 \in H} H(gh_1, h_2|1)\Big) > 0,$$

$$\overline{G}(xh, yh'|z) \quad := \quad \sum_{n=0}^\infty \mathbb{P}_{xh}\big[X_n = yh', \forall m \leq n : \|X_m\| \geq 1\big] z^n.$$

The functions $H(xh, h'|z)$ and $\overline{G}(xh, yh'|z)$ can be computed by solving a finite system of non-linear equations; compare with (2.2) and (2.3). Analogously to Proposition 2.1, it is easy to see that $\big(\mathbf{W}_k, \mathbf{i}_k\big)_{k \in \mathbb{N}}$ is a Markov chain. The state space $\mathcal{Z}$ can now be restricted to

$$\mathcal{Z}_\Gamma := \left\{(xh, n) \,\Big|\, x \in \bigcup_{i=1}^r R_i, h \in H, n \in \mathbb{N}\right\}.$$

Define $[x_1 \ldots x_n h] := x_n h$. Then $([\mathbf{W}_k])_{k \in \mathbb{N}}$ is also a irreducible Markov chain on a finite state space with invariant probability measure ν. Thus, we get

$$\Lambda = \sum_{\substack{xh, yh' \in \bigcup_{i=1}^r \Gamma^\times_i, \\ \tau(xh) \neq \tau(yh')}} \nu(yh') \cdot \frac{\xi\big(\tau(xh)\big)}{\xi\big(\tau(yh')\big)} \cdot \frac{\partial}{\partial z}\left[\sum_{w \in \Gamma^\times_{\tau(x)}} p(yh', yh'w) \cdot z \cdot \overline{G}(h'w, xh|z)\right]\Bigg|_{z=1}$$

and

$$\Delta = \sum_{\substack{xh, yh' \in \bigcup_{i=1}^r \Gamma^\times_i, \\ \tau(xh) \neq \tau(yh')}} \nu(yh') \cdot \frac{\xi\big(\tau(xh)\big)}{\xi\big(\tau(yh')\big)} \cdot l(x) \cdot \sum_{w \in \Gamma^\times_{\tau(x)}} p(yh', yh'w) \cdot \overline{G}(h'w, xh|1).$$

Finally, we obtain:

Corollary 3.2

$$\lim_{n\to\infty} \frac{l(X_n)}{n} = \frac{\Delta}{\Lambda} \quad \textit{almost surely.}$$

3.3 Computation by Double Generating Functions

In this section we derive another formula for the rate of escape with the help of a theorem of Sawyer and Steger (21, Theorem 2.2), which we reformulate adapted to our situation:

Theorem 3.3 (Sawyer and Steger) *Suppose we can write for some $\delta > 0$*

$$\mathcal{E}(w,z) := \mathbb{E}\left(\sum_{n\geq 0} w^{l(X_n)} z^n\right) = \frac{C(w,z)}{g(w,z)} \quad \textit{for } w,z \in (1-\delta;1),$$

where $C(w,z)$ and $g(w,z)$ are analytic for $|w-1|, |z-1| < \delta$ and $C(1,1) \neq 0$. Then

$$\frac{l(X_n)}{n} \xrightarrow{n\to\infty} \ell = \frac{\frac{\partial}{\partial w}g(1,1)}{\frac{\partial}{\partial z}g(1,1)} \quad \textit{almost surely.}$$

Moreover, if $(X_n)_{n\in\mathbb{N}_0}$ is a reversible Markov chain, then with $\bar{g}(r,s) := g(e^{-r}, e^{-s})$

$$\frac{Y_n - n\ell}{\sqrt{n}} \xrightarrow{n\to\infty} N(0,\sigma^2) \textit{ in law, where } \sigma^2 = \frac{-\frac{\partial^2}{\partial^2 r}\bar{g}(0,0) + 2\ell\frac{\partial^2}{\partial s\partial r}\bar{g}(0,0) - \ell^2\frac{\partial^2}{\partial^2 s}\bar{g}(0,0)}{\frac{\partial}{\partial s}\bar{g}(0,0)}.$$

We remark that (21, Theorem 2.2) also comprises a central limit theorem. Similar limit theorems are well-known in analytical combinatorics, see e.g. Bender and Richmond (1) and Drmota (4), (5). We show now how to write the expectation in the theorem in the required way. Let s_H be the stopping time of the first return to H after start at e, that is, $s_H = \inf\{1 \leq m \in \mathbb{N} \mid X_m \in H\}$. For $h \in H$, $i \in \{1, \ldots, r\}$, $x \in \Gamma_i \setminus H$ and $z \in \mathbb{C}$ we define

$$L(h,x|z) := \sum_{n\geq 0} \mathbb{P}_h\big[X_n = x, s_H > n\big] z^n = \sum_{y\in\Gamma_i^\times} p(h,y) \cdot z \cdot \overline{G}(y,x|z).$$

Additionally, we set $L(h,h|z) := 1$ and $L(h,h'|z) := 0$ for $h' \in H \setminus \{h\}$. With this notation we have

$$\mathcal{E}(w,z) = \sum_{x\in\Gamma} \sum_{n\in\mathbb{N}_0} p^{(n)}(e,x) z^n w^{l(x)} = \sum_{x\in\Gamma} \sum_{h\in H} G(e,h|z) L(h,x|z) w^{l(x)}.$$

Setting

$$\mathcal{L}_i^+(w,z) \quad := \quad \sum_{x\in\Gamma_i^\times} L(e,x|z) w^{l(x)} \quad \text{and}$$

$$\mathcal{L}_i(w,z) \quad := \quad \sum_{n\geq 1} \sum_{\substack{x_1\ldots x_n h\in\Gamma, \\ x_1\in\Gamma_i^\times}} L(e, x_1\ldots x_n h|z) w^{l(x_1\ldots x_n h)},$$

we have

$$\mathcal{L}(w,z) := \sum_{x\in\Gamma} L(e,x|z)\, w^{l(x)} = 1 + \sum_{i=1}^{r} \mathcal{L}_i(w,z). \tag{3.2}$$

We now rewrite $\mathcal{L}_i(w,z)$:

$$\begin{aligned}
\mathcal{L}_i(w,z) &= \mathcal{L}_i^+(w,z)\cdot\left(1 + \sum_{n\geq 2}\sum_{\substack{x_2\ldots x_n h\in\Gamma\backslash H,\\ x_2\notin\Gamma_1}} L(e,x_2\ldots x_n h|z)\, w^{l(x_2\ldots x_n h)}\right)\\
&= \mathcal{L}_i^+(w,z)\cdot\left(1 + \sum_{j=1,j\neq i}^{r}\mathcal{L}_j(w,z)\right) = \mathcal{L}_i^+(w,z)\cdot\Big(\mathcal{L}(w,z) - \mathcal{L}_i(w,z)\Big). \tag{3.3}
\end{aligned}$$

From (3.2) and (3.3) we obtain

$$\mathcal{L}(w,z) = 1 + \sum_{i=1}^{r}\frac{\mathcal{L}_i^+(w,z)\mathcal{L}(w,z)}{1+\mathcal{L}_i^+(w,z)},$$

yielding

$$\mathcal{L}(w,z) = \frac{1}{1 - \sum_{i=1}^{r}\frac{\mathcal{L}_i^+(w,z)}{1+\mathcal{L}_i^+(w,z)}}.$$

Now we can write the expectation of Theorem 3.3 in the requested way:

$$\begin{aligned}
\mathcal{E}(w,z) &= \sum_{h\in H} G(e,h|z)\sum_{x\in\Gamma} L(e,h^{-1}x|z)\, w^{l(x)}\\
&= \sum_{h\in H} G(e,h|z)\sum_{x\in\Gamma} L(e,x|z)\, w^{l(x)} = \frac{\sum_{h\in H} G(e,h|z)}{1 - \sum_{i=1}^{r}\frac{\mathcal{L}_i^+(w,z)}{1+\mathcal{L}_i^+(w,z)}}.
\end{aligned}$$

Thus, we can apply Theorem 3.3 with $C(w,z) = \sum_{h\in H} G(e,h|z)$ and

$$g(w,z) = 1 - \sum_{i=1}^{r}\frac{\mathcal{L}_i^+(w,z)}{1+\mathcal{L}_i^+(w,z)}.$$

Corollary 3.4 *The rate of escape w.r.t. $l(\cdot)$ is*

$$\lim_{n\to\infty}\frac{l(X_n)}{n} = \frac{\Upsilon_1}{\Upsilon_2}\ \textit{almost surely,}$$

where

$$\Upsilon_1 = \sum_{i=1}^{r}\frac{\sum_{x\in\Gamma_i^{\times}} l(x)\, L(e,x|1)}{\left(1 + \sum_{x\in\Gamma_i^{\times}} L(e,x|1)\right)^2}\ \textit{and}\ \Upsilon_2 = \sum_{i=1}^{r}\frac{\sum_{x\in\Gamma_i^{\times}} L'(e,x|1)}{\left(1 + \sum_{x\in\Gamma_i^{\times}} L(e,x|1)\right)^2}.$$

Proof: Computing the derivatives of $g(w,z)$ w.r.t. w and z leads to the proposed formula. $\qquad\square$

3.4 Computation via the Limit Process

In this section we derive another formula for the rate of escape w.r.t. the natural word length $\|\cdot\|$. First,

$$\mathbb{E}[\|X_n\|] = \sum_{\bar{g}\in\Gamma} \|\bar{g}\|\,\mu^{(n)}(\bar{g}) \quad \text{and} \quad \mathbb{E}[\|X_{n+1}\|] = \sum_{g,\bar{g}\in\Gamma} \|g\bar{g}\|\,\mu(g)\,\mu^{(n)}(\bar{g}).$$

Thus, we have

$$\mathbb{E}[\|X_{n+1}\|] - \mathbb{E}[\|X_n\|] = \sum_{g\in\Gamma}\mu(g)\int_\Gamma \big(\|gX_n\| - \|X_n\|\big)\,d\mu^{(n)}.$$

Since $\mathbb{E}[\|X_n\|]/n$ converges to $\ell = \lim_{n\to\infty}\|X_n\|/n$, it is sufficient to prove that this difference of expectations converges; the limit must then equal ℓ. The process $(X_n)_{n\in\mathbb{N}_0}$ converges to some random element X_∞ valued in

$$\Gamma_\infty := \left\{ x_1 x_2 \cdots \in \Gamma^{\mathbb{N}} \,\middle|\, x_i \in \bigcup_{j=1}^r R_j, \tau(x_i)\neq\tau(x_{i+1}) \right\}$$

in the sense that the length of the common prefix of X_n and X_∞ goes to infinity. We denote by $X_\infty^{(1)}$ the first letter of X_∞ and for $g\in\bigcup_{i=1}^r\Gamma_i$ we define

$$Y_g := \lim_{n\to\infty}\|gX_n\| - \|X_n\| = \begin{cases} 1, & \text{if } X_\infty^{(1)}\notin\Gamma_{\tau(g)} \\ -1, & \text{if } X_\infty^{(1)}\in g^{-1}H \\ 0, & \text{otherwise} \end{cases}$$

At this point we need the equation $\|hx\| = \|x\|$ for $h\in H$ and $x\in\Gamma$. This equation is, in general, not satisfied for other length functions. The Green functions $G(x,y) = \sum_{n\geq 0}\mathbb{P}_x[X_n = y]$, where $x,y\in\Gamma_i$ for any $i\in\{1,\ldots,r\}$, satisfy the following linear recursive equations:

$$G(x,y) = \delta_x(y) + \sum_{w\in\Gamma_i}p(x,w)\,G(w,y) + \sum_{\substack{xwh\in\Gamma,\\\|xwh\|=2}}p(x,xwh)\sum_{h'\in H}H(wh,h'|1)\,G(xh',y).$$

This system of Green functions can be solved, when the functions $H(wh,h'|1)$ can be obtained by solving (2.3). We now define

$$\varrho(i) := \mathbb{P}\big[X_\infty^{(1)}\in\Gamma_i\big] = \sum_{h\in H}G(e,h|1)\sum_{g\in\Gamma_i^\times}\mu(g)\cdot\Big(1 - \sum_{h'\in H}H(hg,h'|1)\Big).$$

By transience, $\sum_{i=1}^r\varrho(i) = 1$. Furthermore, $\mathbb{P}[Y_g = 1] = 1 - \varrho\big(\tau(g)\big)$ and

$$\mathbb{P}[Y_g = -1] = \sum_{h\in H}F(e,g^{-1}h)\cdot\big(1 - \varrho\big(\tau(g)\big)\big) = \frac{1 - \varrho\big(\tau(g)\big)}{G(e,e)}\sum_{h\in H}G(e,g^{-1}h).$$

By Lebesgue's Dominated Convergence Theorem,

$$\mathbb{E}\big[\|X_{n+1}\|\big] - \mathbb{E}\big[\|X_n\|\big] \xrightarrow{n\to\infty} \sum_{i=1}^r\sum_{g\in\Gamma_i^\times}\mu(g)\Big(\mathbb{P}[Y_g = 1] - \mathbb{P}[Y_g = -1]\Big).$$

But this limit must be the rate of escape ℓ. Thus:

Corollary 3.5

$$\ell = \lim_{n\to\infty} \frac{\|X_n\|}{n} = \sum_{i=1}^{r} \left[\mu(\Gamma_i^\times)\big(1 - \varrho(i)\big) - \frac{1 - \varrho(i)}{G(e,e)} \sum_{g\in\Gamma_i^\times} \sum_{h\in H} \mu(g)G(e, g^{-1}h) \right].$$

As a final remark observe that the formulas of Corollaries 3.2, 3.4 and 3.5 have complexities in decreasing order: while the computation of the rate of escape by Corollary 3.2 needs three systems of equations to be solved and derivatives to be calculated, the computation by Corollaries 3.4 or 3.5 needs only two systems of equations to be solved, while the formula in Corollary 3.4 deals also with derivatives.

4 Sample Computations

4.1 A Regular Language

Let be $A = \{a, b, c\}$ and we set $l(a) = l(b) = l(c) = 1$. We consider the set $\mathcal{L}$ of all words over the alphabet A, such that in each $w \in \mathcal{L}$ the letter b is the first letter of w or follows after the letter a and the letter c may only appear after the letter b; e.g., $abcaba \in \mathcal{L}$, but $abcba \notin \mathcal{L}$. Consider the random walk on $\mathcal{L}$ given by the following transition probabilities:

$$p(aa, aaa) = \frac{1}{3}, \ p(aa, aab) = \frac{1}{3}, \ p(aa, a) = \frac{1}{3}, \ p(ab, aba) = \frac{1}{6}, \ p(ab, abc) = \frac{1}{3}, \ p(ab, a) = \frac{1}{2},$$
$$p(ba, baa) = \frac{1}{4}, \ p(ba, bca) = \frac{1}{4}, \ p(ba, bab) = \frac{1}{4}, \ p(ba, a) = \frac{1}{4},$$
$$p(bc, bca) = \frac{1}{2}, \ p(bc, a) = \frac{1}{2}, \ p(ca, caa) = \frac{1}{4}, \ p(ca, cab) = \frac{1}{2}, \ p(ca, a) = \frac{1}{4}.$$

Note that it is not necessary to specify any further transition probabilities, as the formula for the rate of escape does not depend on the transition probabilities of the form $\mathbb{P}[X_{n+1} = w' \mid X_n = w]$, where $w \in \{\varepsilon, a, b, c\}$. The system of equations 2.3 is then

$$
\begin{aligned}
H(aa, a|z) &= \frac{z}{3}\big(H(aa, a|z) \cdot H(aa, a|z) + H(ab, a|z) \cdot H(aa, a|z) + 1\big), \\
H(ab, a|z) &= \frac{z}{3}H(bc, a|z) \cdot H(aa, a|z) + \frac{z}{6}H(ba, a|z) \cdot H(aa, a|z) + \frac{z}{2}, \\
H(ba, a|z) &= \frac{z}{4}\big(H(aa, a|z) \cdot H(ba, a|z) + H(ca, a|z) \cdot H(ba, a|z) + H(ab, a|z) \cdot H(ba, a|z) + 1\big), \\
H(bc, a|z) &= \frac{z}{2}H(ca, a|z) \cdot H(ba, a|z) + \frac{z}{2}, \\
H(ca, a|z) &= \frac{z}{4}H(aa, a|z) \cdot H(ca, a|z) + \frac{z}{2}H(ab, a|z) \cdot H(ca, a|z) + \frac{z}{4}.
\end{aligned}
$$

This system in the unknown variables $H(aa, a|z)$, $H(ab, a|z)$, $H(ba, a|z)$, $H(bc, a|z)$ and $H(ca, a|z)$, where z appears as a parameter, can be solved with the help of MATHEMATICA. With these solutions we can compute the modified Green functions $\overline{G}(\cdot, \cdot|z)$ by solving the linear system (2.2). Note that only $\overline{G}(aa, aa|z)$, $\overline{G}(ab, aa|z)$, $\overline{G}(ba, ba|z)$, $\overline{G}(bc, ba|z)$, $\overline{G}(ca, ca|z)$ are non-zero functions. Moreover, we

get

$$\begin{aligned}
\xi(aaa) &= \xi(baa) = \xi(caa) = \frac{1}{3}\big(1 - H(aa, a|1)\big) + \frac{1}{3}\big(1 - H(ab, a|1)\big), \\
\xi(aab) &= \xi(bab) = \xi(cab) = \frac{1}{6}\big(1 - H(ba, a|1)\big) + \frac{1}{3}\big(1 - H(bc, a|1)\big), \\
\xi(aba) &= \frac{1}{4}\big(1 - H(aa, a|1)\big) + \frac{1}{4}\big(1 - H(ca, a|1)\big) + \frac{1}{4}\big(1 - H(ab, a)\big), \\
\xi(abc) &= \frac{1}{2}\big(1 - H(ca, a|1)\big), \ \xi(bca) = \frac{1}{4}\big(1 - H(aa, a|1)\big) + \frac{1}{2}\big(1 - H(ab, a|1)\big).
\end{aligned}$$

Since $\nu(abc) = \sum_{def \in \overline{A}^3} \nu(def)\, \tilde{q}(def, abc)$, we can compute the invariant measure as

$$\nu(aaa) = 0.32475,\ \nu(aab) = 0.13194,\ \nu(aba) = 0.12597,\ \nu(abc) = 0.08021,\ \nu(baa) = 0.05350,$$
$$\nu(bca) = 0.13095,\ \nu(bab) = 0.02174,\ \nu(caa) = 0.07844,\ \nu(cab) = 0.05251.$$

Now we have all necessary ingredients to compute $\Lambda = 3.78507$, and finally we get the rate of escape as $\ell = 0.264196$.

4.2 $\mathbb{Z}/d\mathbb{Z} *_{\mathbb{Z}/2\mathbb{Z}} \mathbb{Z}/d\mathbb{Z}$

Consider the free product by amalgamation $\mathbb{Z}/d\mathbb{Z} *_{\mathbb{Z}/2\mathbb{Z}} \mathbb{Z}/d\mathbb{Z}$, $d \in \mathbb{N}$ even, over the common subgroup $\mathbb{Z}/2\mathbb{Z}$. Suppose that $\mathbb{Z}/d\mathbb{Z}$ is generated by some element a with a^6 equal to the identity. Setting $\mu_1(a) = \mu_2(a) = 1$ and $\alpha_1 = \alpha_2 = 1/2$ we get the following values for the rate of escape ℓ w.r.t. $\|\cdot\|$:

d	6	8	10	12
ℓ	0.24749	0.40859	0.46144	0.47543

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